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Relativistic heat conduction

Y.M. Ali *, L.C. Zhang

School of Aerospace, Mechanical and Mechatronic Engineering, J07, University of Sydney, NSW 2006, Australia

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Abstract

The hyperbolic heat conduction equation (HHCE), which acknowledges the finite speed of heat propagation, is based on microscopic evidence from the kinetic theory and statistical mechanics. However, it was argued that the HHCE could violate the second law of thermodynamics. This paper shows that a HHCE-like equation (RHCE) can be derived directly from the theory of relativity, as a direct consequence of space–time duality, without any consideration of the microstructure of the heat-conducting medium. This approach results in an alternative expression for the heat flux vector that is more compatible with the second law. Therefore, the RHCE brings the classical field theory of heat conduction into agreement with other branches of modern physics.

Keywords: Relativity; Thermodynamics; Entropy; Hyperbolic

I think that theory cannot be fabricated out of the results of observation, but that it can only be invented.

Albert Einstein [1]

1. Introduction

The theory of relativity is often overlooked within the context of classical engineering sciences based on the following argument: Relativity is concerned with objects moving at speeds comparable with the speed of light, while most mechanical systems involve objects moving at speeds negligible compared with the speed of light; therefore, relativistic effects can be ignored. While the above argument is true, it is a very strong and restrictive interpretation of the relativity theory. A weaker interpretation can be proposed: Relativity is concerned with objects that move or propagate at speeds comparable with a *limiting speed* characteristic of the field or medium involved.

For example, electromagnetic objects propagate at speeds comparable with the maximum speed of propagation of an electromagnetic signal (photon), i.e. the speed of light in vacuum. Similarly, cosmological objects move at significant speeds, when compared with the maximum speed of propagation of a gravitational signal (graviton). However, there are physical fields that are less fundamental in nature, for which the limiting speeds of propagation are (numerically) small compared with the speed of light. Yet, they still impose restrictions on the propagation of objects moving across them. For example, the potential field of a pressure wave propagating through a fluid is restricted by the speed of sound in that fluid, especially near a unit Mach number.

* Corresponding author. Fax: +61 2 9351 7060.
E-mail address: yali@aeromech.usyd.edu.au (Y.M. Ali).

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Nomenclature

$A[\ ]$	relativistic operator
c	specific heat
C	speed of heat (second sound)
d	total derivative
D	substantial derivative
dx, dy, dz	spatial distances
$d\tau$	space-like-time distance
f, h	arbitrary functions
$\mathbf{H}$	heat flow vector
i	imaginary transformation
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	spatial unit vectors
k	thermal conductivity
$\mathbf{o}$	time unit vector
$\mathbf{q}$	heat flux vector
s	specific entropy
t	real time
u	specific energy
$\mathbf{U}$	source velocity vector
x, y, z	spatial dimensions

Greek symbols

α	thermal diffusivity
β	arbitrary parameter
∂	partial derivative
θ	temperature
ρ	density
σ	entropy production
τ	space-like-time
τ_0	relaxation time

Other symbols

∇	gradient operator
∇^2	Laplacian operator
$\Box$	quad operator
$\Box^2$	d'Alembertian operator
$\cdot$	vector dot product
$\cap$	intersection operator
$\rightarrow$	implication operator

A similar effect equally applies to heat conduction, which can be viewed as propagation of hypothetical particles (phonons) in a hypothetical gas (historically know as 'Caloric'). The limiting speed on heat propagation (the speed of second sound) has been measured in various media [2–4]. A thermal Mach number has also been reported for heat conduction through solids [5,6]. Furthermore, thermal resonance has been suggested in cases of high frequency periodic thermal loading [7,8].

The Fourier equation of heat conduction is fundamentally wrong because it assumes an infinite speed of propagation of heat, which is physically inadmissible. This anomaly has been (supposedly) overcome by the hyperbolic heat conduction equation (HHCE), which includes a component (presumably) recognising the finite speed of heat signals. The HHCE was developed based on microscopic considerations [2,9,10] that are cumbersome, and seem to violate at least on statement of the second law of thermodynamics [11].

This paper will show that the HHCE can be consistently developed based on a weak interpretation of the theory of relativity, as originally proposed in [12], without any microscopic or material-specific considerations. The following sections will argue that: (1) the HHCE does not really comply with the relativity theory, and (2) that it can violate the second law of thermodynamics. The alternative model aims at overcoming both difficulties, and hopes to help in resolving the existing controversies about the heat equation. The new model of heat conduction will bring it into better agreement with other branches of modern physics.

2. The hyperbolic heat conduction equation

For any rigid stationary material without internal heat generation, energy balance within a control mass can be expressed as [13]:

$$\rho c \frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{q} = 0, \tag{1}$$

where ρ is density, c is specific heat (at a given temperature θ), $\mathbf{q}$ is the thermal heat flux vector, and ∇ is the gradient operator:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{2}$$

If the material is isotropic, homogeneous, and the variation in temperature is so small that material properties can be assumed constant, then substituting Fourier's linear approximation of heat flux,

$$\mathbf{q} = -k\nabla\theta, \tag{3}$$

into Eq. (1) leads to the well-known Fourier equation of heat conduction:

$$\frac{\partial \theta}{\partial t} = \alpha \nabla^2 \theta, \tag{4}$$

where k is thermal conductivity, $\alpha = k/(\rho c)$ is thermal diffusivity, and ∇^2 is the Laplacian operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

Several investigators have argued that Eq. (4) is fundamentally wrong because it assumes an infinite speed of propagation of heat signals [14]. This means that a thermal disturbance can be detected instantaneously at an infinitely far distance from the source, which is physically unacceptable. One of the implications of the theory of relativity is the principle of "no action at a distance" which requires a finite speed of propagation of any signal, and necessitates a time lag between a cause and its effect [15]. Morse and Feshbach [16] recognised this problem, and proposed that Eq. (4) should be modified into a more acceptable (Telegraph) form:

$$\frac{1}{C^2}\frac{\partial^2\theta}{\partial t^2}+\frac{1}{\alpha}\frac{\partial\theta}{\partial t}=\nabla^2\theta. \tag{6}$$

This form is known as Maxwell's equation, because it resembles the equation of propagation of an electromagnetic field, i.e. light. However, in Eq. (6) the speed of heat propagation, C, is not a fundamental property of the field, but is related to the mean free path of gas molecules. The idea is extended to solids, by assuming that heat is conducted by gas-like phonon or electron streams.

Tisza (1938) and Landau (1941) predicted that the speed of heat can be different from the speed of sound and called it the speed of *second sound* [17]. Peshkov [3] found experimentally that in liquid Helium II, it is one order of magnitude less than the speed of sound. There are evidences to suggest the same ratio or more for non-homogeneous solids [4,18].

Eq. (6) is often attributed to Cattaneo [19], Vernotte [20], as well as Chester [2] who considered the case of second sound in solids. Based on the kinetic theory calculations and Boltzmann equations for a rarefied gas [21], they proposed that Fourier's linear heat flux, Eq. (3), should be modified to the form:

$$\tau_0\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}=-k\nabla\theta, \tag{7}$$

where τ_0 is the relaxation time of the heat conducting medium, i.e. free electrons in the case of metals. If we take the gradient of Eq. (7) and add it to Eq. (7) and the time derivative of Eq. (1) and add it to Eq. (1) and eliminate $\mathbf{q}$ and its derivatives, we get Eq. (6), if we let $C^2=\alpha/\tau_0$.

Here, it is important to note that Eqs. (6) and (7) are physically justified based on microscopic aspects of lattice vibration, electrons transport, and their interactions. The relaxation time, τ_0, has its origin from statistical mechanics of the electrons distribution. Consequently, the speed of second sound, C, is merely a collection of various coefficients in the equation, but has no fundamental physical significance similar to that associated with the speed of light.

In other words, while Eq. (6) looks like satisfying the theory of relativity (because it has the *form* of Maxwell's equation), it is not *conceptually* relativistic. The laws of motion employed in the derivation of the HHCE are essentially Newtonian. The expression in Eq. (7), the misleading symbol C, and even Eq. (6), are only first order approximations of some very complex and cumbersome statistical computations. These computations do not consider the theory of relativity at all, and are classic in nature. A relativistic treatment of the problem was attempted by some authors [22–25], by including a relativistic correction due to the speed of the heat-carrying particles, e.g. electrons. Yet, this is followed by classical statistical treatments that are mathematically complex, and lead to Eqs. (6) and (7) as first order approximations. A review of various approaches to deriving Eqs. (6) and (7) can be found in [26].

3. Controversies about the HHCE

This artificiality in the relationship between the HHCE and the relativity theory is not the only serious objection facing that equation. Several investigators, e.g. [27], argued that Eq. (6) is not necessary because, for most practical situations, C^2 is very large compared with α such that the second-order term is negligible, and Eq. (6) will converge quantitatively to Eq. (4). This argument can be challenged on two grounds. Firstly, there are experimental evidences that C is not always very large. A relaxation time up to 10 s has been measured in non-homogeneous materials [18]. Polymeric and vitreous materials are important engineering materials that are almost thermal *insulators*. While experimental data may not be available yet, there is no reason to believe that C, for these materials, should as high as for metallic *conductors*. Other qualitative evidences, Appendix A, also suggest that under conditions prevailing in many manufacturing processes, C can be very small.

Secondly, even if C were very large, this still would not justify dropping the second order term in Eq. (6). For example, there can be transient conditions in which the second order time derivative is much larger than the first order time derivative. Thus, while the relative ratio of C^2 to α is important, what is more important is the period of the thermal load compared with the relaxation time. For example, laser heating uses pulse frequency in the GHz range. This is a time scale comparable with the relaxation time, and Eq. (6) should be used instead of Eq. (4).

Aside from the quantitative, experimental, and practical issues, Eq. (6) has other important physical implications. Eq. (4) is a parabolic partial differential equation, i.e., a diffusion equation that includes a dissipative component, is always stable, and will always converge to steady state conditions after sufficient time [16]. On the other hand, Eq. (6) is a hyperbolic diffusion wave equation that contains a conservative term, which can result in the production and propagation of thermal waves. Depending on boundary conditions, these waves may be overdamped or underdamped [7,10]. Consequently, there will be cases where initial and boundary conditions may lead to thermal resonance [7,8], or temperature may overshoot to values higher than at the source. Moreover, there can be significant phase differences among various parts of the medium, if subjected to high thermal frequency [28]. Clearly, all these aspects are of practical importance, and the HHCE should be considered whenever fast moving heat sources are involved.

Finally, as noted by many authors [11,29,30], it is possible to design boundary conditions such that heat would appear to be moving from a cold to a hot point, in violation of the second law of thermodynamics. The following section will show that this paradoxical situation can occur not because of Eq. (6) but due to Eq. (7), which has not been verified.

4. Relativistic heat conduction equation

In classical physics, real world is made of a three-dimensional Euclidean space (x, y, z in Cartesian coordinates) and a one-dimensional time (t). Physical laws are required to be invariant with respect to any Galilean transformation. In other words, physical quantities remain the same for all inertial frames of reference within a Euclidean space. For example, the distance, ds, between any two points, as defined by

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{8}$$

remains invariant regardless of any rotation or translation of the coordinate system.

In relativistic physics, a Minkowski world [15] is made of a four-dimensional pseudo-Euclidean space–time (τ, x, y, z), where τ is called "Space-like-time": it has a length dimension. In other words, space and time are coupled, and time is no longer invariant and independent of space. This is a direct consequence of the realization that there is a finite speed of propagation of information, and therefore time does not have the same meaning at all points in space. In the context of heat transfer, this has the following implication. Someone located at point (x, y, z) can solve a Fourier equation and obtain results different from those obtained by someone else at (x', y', z'), and yet both solution can be correct. This is because the first solves the problem according to the information available at time t in its frame of reference, while the other solves the problem at time t' in its own frame of reference. In a Minkowski world, t and t' are not identical, because of the finite speed of propagation of information from and to each of the respective points.

In a Minkowski world, laws of physics must be invariant with respect to a Lorentz transformation. In other words, the interval, ds, between any two events, as defined by

$$ds^2 = dx^2 + dy^2 + dz^2 - C^2 dt^2 \tag{9}$$

remains invariant in all inertial frames of reference. Comparison between Eqs. (8) and (9) shows that any classical physical quantity can be upgraded into its relativistic counterpart by performing an *imaginary rotation* from real time to space-like-time [16], i.e.

$$\tau = iCt, \tag{10}$$

where τ has a length dimension, t has a time dimension, and therefore C has a velocity dimension. Multiplication by the imaginary constant, $i = \sqrt{-1}$, ensures that τ remains orthogonal to the other three spatial dimensions.

Eq. (10) provides a simple mechanism for transforming any classical physical quantity into its relativistic counterpart. For example, the four-dimensional gradient (also called the quad, $\Box$) is defined as [16]:

$$\begin{aligned}\Box &= \frac{\partial}{\partial \tau}\mathbf{o} + \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \\ &= \frac{\partial}{\partial \tau}\mathbf{o} + \nabla \\ &= \frac{-i}{C}\frac{\partial}{\partial t}\mathbf{o} + \nabla.\end{aligned} \tag{11}$$

Similarly, the four-dimensional Laplacian (also called the d'Alembertian, $\Box^2$) is

$$\begin{aligned}\Box^2 &= \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial \tau^2} + \nabla^2 \\ &= \frac{-1}{C^2}\frac{\partial^2}{\partial t^2} + \nabla^2.\end{aligned} \tag{12}$$

The operators in Eqs. (11) and (12) can then be used to upgrade Fourier equation into its relativistic form. However, it is important to note that in the above development and in Eq. (10), there were no pre-assumptions or restrictions on the numeric value of C. In this weak interpretation of the relativity theory, we would like to maintain this ambiguity. Indeed, we would suggest that C is the speed of propagation of information about temperature in the conducting medium. Whether this information is carried by electrons, phonons, or any other mechanism is irrelevant to the current development.

With reference to Fourier equation, Eq. (4), and the definition of a d'Alembertian, Eq. (12), the relativistic Fourier equation in a four-dimensional space–time is

$$\frac{\partial \theta}{\partial t} = \alpha \Box^2 \theta = \alpha\left(\frac{\partial^2 \theta}{\partial \tau^2} + \nabla^2 \theta\right) = \frac{-\alpha}{C^2}\frac{\partial^2 \theta}{\partial t^2} + \alpha \nabla^2 \theta, \tag{13}$$

which is identical in *form* to the hyperbolic heat equation, Eq. (6). However, it is important to note that the physical interpretation and conceptual origin of the parameter C

are very different between the two equations. In Eq. (6), C is just a collection of terms that result from an approximation of the statistical mechanics of diffusion of a gas. In Eq. (13), by virtue of Eq. (10), C is a fundamental property of space–time as seen by the thermal field. In other words, while Eq. (6) is a statistical constitutive equation, Eq. (13) is a field equation describing the propagation of heat, even in vacuum. Moreover, in deriving Eq. (13), we need not know anything about the microstructure of the heat-conducting medium.

In order to appreciate why the HHCE, Eq. (6), can violate the second law of thermodynamics while Eq. (13), which is apparently identical, does not, we need to complete the development of the relativistic model. The energy balance, Eq. (1), when upgraded to the relativistic form, becomes:

$$\rho c\frac{\partial\theta}{\partial t}+\Box\cdot\mathbf{q}=\rho c\frac{\partial\theta}{\partial t}+\frac{-\mathrm{i}}{C}\frac{\partial\mathbf{q}}{\partial t}\cdot\mathbf{o}+\nabla\cdot\mathbf{q}=0, \tag{14}$$

while the definition of the heat flux vector, $\mathbf{q}$ in Eq. (3), in relativistic form, is

$$\mathbf{q}=-k\Box\theta=\frac{\mathrm{i}k}{C}\frac{\partial\theta}{\partial t}\mathbf{o}-k\nabla\theta. \tag{15}$$

It can be seen that substituting Eq. (15) into Eq. (14) yields Eq. (13), and that the energy balance equation is preserved in both worlds. The *imaginary* component in Eq. (15) is a straightforward manifestation of the wave nature of heat conduction that is implied in Eq. (13). Therefore, the relativistic heat conduction model in four-dimensions,[1]

$$\rho c\frac{\partial\theta}{\partial t}+\Box\cdot\mathbf{q}=0\ \cap\ \mathbf{q}=-k\Box\theta\ \rightarrow\ \frac{\partial\theta}{\partial t}=\alpha\Box^2\theta, \tag{16}$$

is consistent, and has identical form to the classical (three-dimensional) Fourier model,

$$\rho c\frac{\partial\theta}{\partial t}+\nabla\cdot\mathbf{q}=0\ \cap\ \mathbf{q}=-k\nabla\theta\ \rightarrow\ \frac{\partial\theta}{\partial t}=\alpha\nabla^2\theta. \tag{17}$$

Consequently, one would argue that if the system of Eq. (17) does not violate the laws of thermodynamics, then the system of Eq. (16) should not violate them as well.

The reason why the Maxwell–Chester–Cattaneo–Vernotte model of heat conduction (the set of Eqs. (1), (7), and (6)) violates the laws of thermodynamics now becomes clearer. This model has the structure:

$$\rho c\frac{\partial\theta}{\partial t}+\nabla\cdot\mathbf{q}=0\ \cap\ \tau_0\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}=-k\nabla\theta\ \rightarrow\ \frac{\partial\theta}{\partial t}=\alpha\Box^2\theta. \tag{18}$$

It can be seen that the temperature equation, Eq. (6), is the HHCE, which is identical to the relativistic form in Eq. (13). The energy equation, Eq. (1), is the same as that for the classical (three-dimensional) Fourier model, Eq. (17). The heat flux equation, Eq. (7), however, is a confused formulation that belongs to neither world. The model in Eq. (18) is neither classical nor relativistic, but something in between, mainly due to the structure of Eq. (7). It is interesting to note that Eckert and Drake have cautioned against the use of Eq. (7), "as it has not been established neither experimentally or theoretically" [14]. A careful inspection of Eq. (7) would suggest that the time derivative is a form of *heat leakage* along the time direction in Eq. (3). As this leakage is unaccounted for, there may be situations in which Eq. (18) can violate the laws of thermodynamics.

To appreciate further the difference between the systems of Eq. (16) and Eq. (18), and why Eq. (7) is invalid while Eq. (15) is fine, take the time derivative of Eq. (15):

$$\frac{\partial\mathbf{q}}{\partial t}=\frac{\mathrm{i}k}{C}\frac{\partial^2\theta}{\partial t^2}\mathbf{o}-k\nabla\left(\frac{\partial\theta}{\partial t}\right). \tag{19}$$

Then, we can write

$$\tau_0\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}=\frac{\mathrm{i}k}{C}\left[\tau_0\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}\right]\mathbf{o}-k\left[\tau_0\nabla\left(\frac{\partial\theta}{\partial t}\right)+\nabla\theta\right], \tag{20}$$

which is significantly different from Eq. (7). Even if we consider only the real part,

$$\mathrm{Re}\left[\tau_0\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}\right]=-k\nabla\left[\tau_0\frac{\partial\theta}{\partial t}+\theta\right], \tag{21}$$

it has a component of temperature time derivative that is missing in Eq. (7). In a four-dimensional Minkowski world, Eq. (20) can be written as:

$$\left[\tau_0\frac{\partial\mathbf{q}}{\partial t}+\mathbf{q}\right]=-k\Box\left[\tau_0\frac{\partial\theta}{\partial t}+\theta\right]. \tag{22}$$

Hence, it is clear that Eqs. (20)–(22) are symmetric, while Eq. (7) is not.

Symmetry of the form of physical laws is often taken as an evidence of the validity of these laws.

5. Thermodynamic considerations

The classical local entropy balance over a control mass is [11]:

$$\nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right)+\rho\frac{\partial s}{\partial t}=\sigma, \tag{23}$$

where s is specific entropy and σ is entropy production rate. Combining Eq. (23) with Eq. (1) and the Gibbs equation,

$$c\,\mathrm{d}\theta=\mathrm{d}u=\theta\,\mathrm{d}s, \tag{24}$$

[1] In Eqs. (16)–(18), "∩" means "and," while "→" means "therefore".

leads to

$$\sigma = \frac{-1}{\theta^2}\mathbf{q}\cdot\nabla\theta. \tag{25}$$

One statement of the second law of thermodynamics is the Clausius inequality, which requires that σ should remain non-negative for a closed system, changing irreversibly between two equilibrium states. Firstly, Eq. (6) is a non-equilibrium equation, and there is no universally acceptable non-equilibrium thermodynamics theory [11,31]. Indeed, the concepts of temperature and entropy are not clearly defined under non-equilibrium conditions. Secondly, while Eq. (25) applies for a system, we adopt a more strict statement and require that Eq. (25) applies at each point in space and time. Therefore, violating Eq. (25) does not necessarily mean violating a non-equilibrium second law, but an equation satisfying Eq. (25) is likely to satisfy a more general non-equilibrium second law, if such a law can be established.

Substituting Fourier's definition of heat flux, Eq. (3), into Eq. (25) yields:

$$\sigma = \frac{k}{\theta^2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2\right], \tag{26}$$

which is always non-negative. Therefore, Fourier equation, Eq. (4), and the definition of heat flux, Eq. (3), are always in agreement with the first, Eq. (1), and second, Eq. (23), laws of thermodynamics, because Eq. (26) is always non-negative.

On the other hand, the Cattaneo-Vernotte definition of heat flux, Eq. (7), when substituted into Eq. (25), leads to [11]:

$$\sigma = \frac{1}{k\theta^2}\left(\mathbf{q}\cdot\mathbf{q} + \tau_0\mathbf{q}\cdot\frac{\partial\mathbf{q}}{\partial t}\right), \tag{27}$$

which cannot be reduced to a relation in θ alone, such as in Eq. (26), due to the anomalous presence of the time derivative of $\mathbf{q}$. Clearly, there is no guarantee that Eq. (27) is always non-negative; there may be conditions in which Eq. (27), and consequently Eq. (7), can violate the second law of thermodynamics [11]. For example, with very large relaxation time and very large negative rate of change of the heat flux vector, the second term in Eq. (27) can become negative and larger than the first term; thus, implying negative entropy production.

Finally, when the relativistic heat flux, Eq. (15), is combined with the relativistic energy balance in Eq. (14), the Gibbs relation in Eq. (24), and the relativistic entropy balance,

$$\Box\cdot\left(\frac{\mathbf{q}}{\theta}\right) + \rho\frac{\partial s}{\partial t} = \sigma, \tag{28}$$

it yields:

$$\sigma = \frac{-1}{\theta^2}\mathbf{q}\cdot\Box\theta. \tag{29}$$

Eq. (29), due to the good nature of Eq. (15), can be expanded as

$$\sigma = \frac{k}{\theta^2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 + \left(\frac{\partial\theta}{\partial\tau}\right)^2\right], \tag{30}$$

which is always non-negative in a Minkowski world, and even in real world. To see that this is indeed the case, we note that $\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}x}$, etc., and rewrite Eq. (30) with respect to real time as

$$\sigma = k\left(\frac{1}{\theta}\frac{\partial\theta}{\partial t}\right)^2\left[\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}t}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 - \left(\frac{1}{C}\right)^2\right], \tag{31}$$

which is assured to be non-negative if

$$\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}t}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 \geqslant \frac{1}{C^2}. \tag{32}$$

This inequality is a statement of the second law of thermodynamics that includes neither entropy nor temperature. Indeed, Eq. (32) is a statement of a fundamental property of space–time, regardless of the field under consideration.

The inequality in Eq. (32) is always valid because it is a direct consequence of the definition of a Minkowski world, Eq. (9). To appreciate this point, consider, for simplicity, a two-dimensional space–time (τ,x). For any two time-like unique events, $\mathrm{d}s^2 < 0$, Eq. (9) leads to the conclusion that $C\mathrm{d}t > \mathrm{d}x$. This can only occur if the magnitude of the speed of propagation is less than the magnitude of the limiting speed, C. This is also one of the pillars of the relativity theory, i.e. only time-like events are physically admissible. For example, the speed of any propagation cannot exceed the speed of light in vacuum, by definition of a Minkowski world, Eq. (9). Therefore, if the speed of propagation is less than C, the reciprocal of that speed has to be greater than $1/C$, which is the condition in Eq. (32).

In other words, for the HHCE to satisfy the second law of thermodynamics, it suffices to have a definition of the heat flux vector that is invariant with respect to a Lorentz transformation. The heat flux model in Eq. (15) is invariant with respect to such a transformation, while Eq. (7) is not. Invariance with respect to a Lorentz transformation seems to result in an automatic satisfaction of the second law of thermodynamics. In a sense, the second law becomes nothing but a complementary definition of a Minkowski world, in which no propagation is allowed to exceed the limiting speed, C. Therefore, the conceptual superiority of Eq. (15) over Eq. (7) is asserted.

In conclusion, the technical importance of the HHCE is preserved, while its apparent anomalies have been removed. The proper model to use is therefore the one in Eq. (16), which has been achieved as a direct implication of Eq. (10). The HHCE, Eq. (6), is physically sound, but the heat flux definition in Eq. (7) is not. A heat flux definition according to Eq. (15) seems to be more consistent with the second law and the theory of relativity. To avoid confusion between Eqs. (13) and (6), the earlier will be called the Relativistic Heat Conduction Equation (RHCE). They both have the same form, but have significantly different underlying assumptions and physical backgrounds. In particular, they have different definitions of the heat flux vector, as can be seen by comparing Eqs. (16) and (18).

6. Alternative formulations

When substituting Eq. (3) into Eq. (1), it is common to eliminate $\mathbf{q}$ to yield the Fourier equation, Eq. (4), which is written in terms of θ alone. It is also possible to eliminate θ and write the classical heat conduction equation in terms of $\mathbf{q}$ alone, i.e.

$$\frac{\partial \mathbf{q}}{\partial t} = \alpha \nabla^2 \mathbf{q}. \tag{33}$$

Likewise, if we take the quad of Eq. (14),

$$\rho c \Box \left(\frac{\partial \theta}{\partial t} \right) + \Box^2 \cdot \mathbf{q} = 0, \tag{34}$$

and the time derivative of Eq. (15),

$$\frac{\partial \mathbf{q}}{\partial t} = -k \Box \left(\frac{\partial \theta}{\partial t} \right), \tag{35}$$

then the relativistic heat conduction equation can be written as

$$\frac{\partial \mathbf{q}}{\partial t} = \alpha \Box^2 \mathbf{q}. \tag{36}$$

This formulation is more suitable for problems with heat flux boundary conditions.

It is also possible to define a heat-flow vector field, $\mathbf{H}$, such that:

$$\frac{\partial \mathbf{H}}{\partial t} = \mathbf{q}, \tag{37}$$

and therefore,

$$\rho c \theta = -\Box \cdot \mathbf{H}. \tag{38}$$

This formulation can be useful in that it eliminates the time derivative of temperature, and can be mathematically easier to solve. It also allows for a range of integral, Lagrangian, and variational formulations, which can be more useful for numerical solutions. The transition from Eq. (38) to any of these formulations follows the same steps as the classical procedures described in [14], and need not be repeated here.

Moreover, we can define the relativistic operator, $A[\ ]$, as:

$$A[f] = \tau_0 \frac{\partial f}{\partial t} + f. \tag{39}$$

Then, the relativistic heat flux vector, in Eq. (15), (20), or (22), can be written in a symmetric form as

$$\frac{1}{k} A[\mathbf{q}] = A[\Box \theta] = \Box A[\theta] = \frac{\mathrm{i}}{C} A\left[\frac{\partial \theta}{\partial t} \right] - A[\nabla \theta]. \tag{40}$$

Consequently, the relativistic heat conduction equation can be written as

$$\frac{\partial A[\theta]}{\partial t} = \alpha \nabla^2 \theta, \tag{41}$$

which has identical form to the classical Fourier equation, if θ is replaced with $A[\theta]$ in the time derivative.

Finally, the relativistic heat conduction model, Eq. (16), can be formulated in cylindrical, spherical, or any other coordinate system, by virtue of Eq. (11) and Eq. (12), without any conceptual difficulties. Anisotropic material behaviour can be treated as in the classical case (e.g. [14]), as long as C remains constant. When C is constant, time is isotropic, and only spatial anisotropy needs to be considered. Treatment of temperature dependent material properties, e.g., variable thermal conductivity, can be done by substitution of variables (e.g. [13]) and represents no fundamental difficulties.

7. Restriction

In the above formulations of Section 6, it was implicitly assumed that space and time differentiations are interchangeable, i.e.

$$\frac{\partial}{\partial t} \nabla = \nabla \frac{\partial}{\partial t}, \tag{42}$$

and

$$\frac{\partial}{\partial t} \Box = \Box \frac{\partial}{\partial t}. \tag{43}$$

In classical heat conduction, Eq. (42) is probably acceptable, because it is assumed that space and time are independent and separable. This is not the case with relativity, where space and time are coupled, and there may be circumstances at which Eq. (43) may not hold. This is because real-time and space-like-time are linked, due to Eq. (10). In this weak interpretation of relativity, C is no longer a universal constant; it is a material property that varies with temperature, the presence of stress–strain fields, as well as electro-magnetic fields. This means that C can have sharp local variations depending on external loading as well as the heat conduction field itself. This may result in distortion and wrinkles in space–time itself, and Eq. (43) may not be that straightforward. Thus, care has to be taken when using any of the formulations in Section 6. It is always better to use the original model, Eq. (16), that does not require Eq. (43).

In summary, as long as C is well behaved, in the sense of being practically constant, the relativistic heat conduction model presented here can utilize all of the resources and techniques available to classical heat conduction based on Fourier equation. On the other hand, if C is ill behaved, there may be unforseen consequences that are beyond the scope of this work.

8. Relativistic moving heat source

There are increasing number of practical applications and manufacturing processes in which the RHCE seems important. Many of these cases involve a heat source moving relative to the surface of the work-material being processed; e.g. continuous annealing, laser cutting, welding, high-speed machining, and high-speed grinding. As demand for higher production rates increase, speeds of moving heat sources also increase. In cases like pulsed laser applications and high-speed grinding, the heat sources can have very high frequencies, as well. Thus, temporal speed becomes as important as spatial speed, and the relativistic heat model needs to be considered.

In this section, we will develop the equation of a relativistic moving heat source from basic principles. Consider a four-dimensional pseudo-Euclidean (Minkowski) world defined by the Cartesian coordinates (τ, x, y, z). Consider a four-dimensional hyper-cube located at the origin with dimensions $2\mathrm{d}\tau$, $2\mathrm{d}x$, $2\mathrm{d}y$, $2\mathrm{d}z$. The rate of heat flow across the face $(x - \mathrm{d}x)$ perpendicular to the x-axis is

$$8\left(q_x - \frac{\partial q_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}z, \tag{44}$$

while heat flow across the $(x + \mathrm{d}x)$ face is

$$8\left(q_x + \frac{\partial q_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}z. \tag{45}$$

Therefore, the net heat flow along the x-axis is

$$16\frac{\partial q_x}{\partial x}\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{46}$$

A similar expression can be obtained for each of the other *three* axes. Meanwhile, heat accumulated inside the hyper-cube, in the absence of internal heat generation, is

$$16\rho c\frac{\partial \theta}{\partial t}\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{47}$$

Consequently, the equation for energy balance is

$$\rho c\frac{\partial \theta}{\partial t} + \left(\frac{\partial q_\tau}{\partial \tau} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) = 0. \tag{48}$$

In case of a control volume that is moving with speeds $\mathbf{U} = (0, U_x, U_y, U_z)$ relative to the stationary frame of reference, there are convective heat components $(0, U_x\rho c\theta, U_y\rho c\theta, U_z\rho c\theta)$. The expression for relativistic heat flux, Eq. (15), becomes:

$$\mathbf{q} = -k\Box\theta + \rho c\theta\mathbf{U}. \tag{49}$$

When Eq. (49) is substituted into Eq. (48), and assuming a homogeneous isotropic material with constant material properties, we get

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \alpha\Box^2\theta, \tag{50}$$

where D is the substantial derivative:

$$\begin{aligned}\frac{D\theta}{Dt} &= \frac{\partial \theta}{\partial t} + 0\frac{\partial \theta}{\partial \tau} + U_x\frac{\partial \theta}{\partial x} + U_y\frac{\partial \theta}{\partial y} + U_z\frac{\partial \theta}{\partial z}\\ &= \frac{\partial \theta}{\partial t} + U_x\frac{\partial \theta}{\partial x} + U_y\frac{\partial \theta}{\partial y} + U_z\frac{\partial \theta}{\partial z}.\end{aligned} \tag{51}$$

It is important to note that in Eq. (51), $\mathbf{U}$ is the velocity of the moving continuum relative to a stationary coordinate system. Of course, if the coordinate system is moving relative to a stationary medium, we only need to reverse the sign from $\mathbf{U}$ to $-\mathbf{U}$. Moreover, it is always possible to align the coordinate system with the velocity vector along the x-axis, such that $\mathbf{U} = U_x$, and Eq. (50) simplifies to

$$\frac{\partial \theta}{\partial t} + U_x\frac{\partial \theta}{\partial x} = \alpha\Box^2\theta. \tag{52}$$

Eq. (52) will be used in a related work [32] to demonstrate the practical applications of the relativistic heat conduction model presented here.

9. Quantum mechanics relations

As shown in [32], after some dimensional manipulation of a function of temperature, $h(\theta)$, Eq. (52) can be written as:

$$\Box^2 h(\theta) - \beta^2 h(\theta) = 0, \tag{53}$$

where β is a function of velocity, U_x, alone. Eq. (53) is similar in form to the homogeneous *Klein–Gordon* equation that arises in relativistic quantum mechanics of a "scalar" meson or an elementary particle without spin [16]. Moreover, by a procedure similar to that used in obtaining Eq. (36), we can also write Eq. (53) in terms of a function of the heat flux vector $h(\mathbf{q})$. That turns out to be similar to the *Proca* equation for a relativistic hypothetical elementary particle with unit spin [16]. Indeed, the heat flux vector and temperature form a complex pair that satisfies both Proca and Klein–Gordon equations, respectively. Therefore, one may speculate that heat is transported by an elementary particle of some peculiar spin characteristics.

Finally, since the RHCE is, after all, a wave equation, it should be possible to find a Ψ-function for the heat wave that satisfies some form of a Schrödinger equation. All of these developments are way beyond the confines of a continuum field theory. They are also beyond the practical interest of the present work, and will not be pursued any further. They are presented here to demonstrate the importance of the new definition of the heat flux vector, Eq. (15), in assimilating classical and quantum physics.

In short, the relativistic heat conduction model, Eq. (16), not only satisfies classical mechanics, thermodynamics, and relativistic mechanics, but also is more compatible with quantum mechanics and electrodynamics. The advantage of this model over the model in Eq. (18) is clearly demonstrated.

10. Conclusion

Fourier equation of heat conduction satisfies both the first and second laws of thermodynamics but is incompatible with the relativity theory. The hyperbolic heat conduction equation, HHCE, is compatible (in form but not in concept) with the theory of relativity and the first law of thermodynamics, but may violate at least one statement of the second law.

This paper has developed an alternative model for heat conduction, based on a weak interpretation of the theory of relativity. This is achieved by relaxing the definition of C from speed of light in vacuum to some *limiting speed* of the medium. The value of C is no longer a universal constant, nor does it have to be numerically very large. It has been shown that such an approach would lead to a new definition of the heat flux vector, **q**, which would satisfy the second law of thermodynamics. The resulting RHCE is shown to satisfy all laws of physics, if it remains invariant with respect to a Lorentz transformation, which is achieved by the new definition of **q**. The relativistic heat conduction model, Eq. (16), is consistent with all laws of physics, and is a more accurate representation of heat conduction in many technologically important situations, to be discussed elsewhere. Moreover, it restores symmetry and a sense of elegance to the equations governing this important branch of physics.

The achievements claimed by this paper are:

1. Proper derivation of the heat equation from relativity theory.
2. Reconcilement with the second law of thermodynamics.
3. A new definition of the heat flux vector.
4. A general field theory of heat conduction, not a statistical constitutive model.
5. New interpretation of the theory of relativity and the space–time conversion factor, C.
6. Assimilation with quantum mechanics and electrodynamics.

Appendix A. On the speed of heat propagation

The present disadvantage is that there are no experimental methods for the accurate determination of C under typical conditions of very high stresses, strains, and strain rates, which prevail in many manufacturing processes. The following qualitative points are intended to show that C may be very small under those conditions:

1. Macroscopically, the equation of speed of propagation of elastic (sound) and plastic waves [33,34] shows that:
$$C^2 = \frac{1}{\rho}\frac{\partial s}{\partial e}, \tag{54}$$
where ρ is density, s and e are engineering stress and strain, respectively. Speed of propagation is clearly proportional to the slope of the stress–strain curve. For most materials, this slope approaches zero as strain and strain rate approach infinity. Therefore, at extremely high strains and strain rates (common in many manufacturing processes), speed of propagation of any wave, including heat, is virtually zero.
2. Microscopically, severe plastic deformation is propagated by heavy lattice imperfections known as dislocations. It is known that dislocations impose significant drag on phonons and electrons and lead to relativistic effects at very high strain rates [35]. Therefore, it is expected that the speed of heat can be very small in the presence of severe plastic deformation. The extremely high strains and strain rates provide conditions in which the material is, microscopically, highly non-homogeneous, with heat "particles" finding it very hard to move.
3. Experimentally, it is established that the speed of heat (second sound) is about one order of magnitude less than the speed of sound [3]. Therefore, if speed of sound in metals is several km/s, the speed of heat can be several hundred m/s, which is not very high compared with speeds prevailing in processes such as high-speed machining and grinding. Moreover, the above values are for metals and perfect structures. Sound and heat are propagated by mechanisms such as lattice vibration, phonons, and electrons, which are all decelerated by structural imperfections. For example, measurements within inhomogeneous materials reported very large relaxation times, and consequently very small speeds of heat [4,18].
4. Modern high speed machining processes involve heat sources moving as high as 1200 m/s [33], i.e. supersonic. Even if speed of heat were as high as speed of sound, the relativistic effect would still be very important.
5. As shown in [32], the finite speed of heat is effectively equivalent to an adiabatic boundary moving away from the heat source, with heat trapped behind that adiabatic front. This effect is observed experimentally in what is know as "adiabatic shear bands." Under the conditions of high strains and strain rates inside the shear zone, temperature rises very sharply in a manner that can be explained only by the presence of an adiabatic boundary around the shear zone

[36]. In other words, inside the shear band, severe plastic deformation drags the heat carriers to the extent that they become unable to propagate far away from the heat source. This is demonstrated by very sharp temperature gradients and consequent localised microstructural changes. The presence of adiabatic shear bands is clear evidence that C was very small and that the relativistic heat model should apply.

In conclusion, while experimental measurements are difficult at present, qualitative analysis would suggest that C might be very small under conditions of high strain and strain rate. The value of C can be much smaller in the case of non-homogeneous or thermally insulating material. In all cases, speed of heat is at least one order of magnitude less than speed of sound, which in itself is not very large in comparison with the heat source velocities applied by many manufacturing processes. Side effects, such as adiabatic shear bands, serve as supporting evidences that the relativistic effect is taking place. Thus, the RHCE should be used in any modelling or analysis of this sort of manufacturing processes.

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